Physics (H) SEM_IV CC VIII (MATHEMATICAL PHYSICS-III: Complex Analysis)

In our last class we have studied:

Brief Revision of Complex Numbers and their Graphical Representation. Euler's formula, De Moivre's theorem

Today we shall learn the following topic: Roots of Complex Numbers.

In our last class we have used De Moivre's Theorem to find out the value of any complex number (z) raised to any power n , i.e., $(z^n)=[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

In today's class we shall use De Moivre's Theorem to find out a general formula for finding the nth roots of a nonzero complex number.

Suppose that we have a complex number $z = r(\cos\theta + i\sin\theta)$. Let the nth root of z be another complex number ρ, then, $\rho^n = z = r(\cos\theta + i\sin\theta)$--------------------------------------- (eq.1)

Or, $\rho = \sqrt[n]{z} = z^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}} = \sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$,

The above expression for ρ is one of the n roots of (eq.1) as it is an equation of degree n. So we have to find out a general expression for all the n roots and it will be,

$$\rho = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right) = \sqrt[n]{r}e^{i(\frac{\theta+2k\pi}{n})},$$

Where, $k = 0, 1, 2, \ldots\ldots\ldots, (n-1)$. Hence, different roots of the (eq.1) would be,

$$\rho_1 = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) = \sqrt[n]{r}e^{i(\frac{\theta}{n})}\ ;\ (\text{putting } k = 0),$$

$$\rho_2 = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right) = \sqrt[n]{r}e^{i(\frac{\theta+2\pi}{n})};\ (\text{putting } k = 1)$$

$$\rho_3 = \sqrt[n]{r}\left(\cos\frac{\theta+4\pi}{n} + i\sin\frac{\theta+4\pi}{n}\right) = \sqrt[n]{r}e^{i(\frac{\theta+4\pi}{n})};\ (\text{putting } k = 2)$$

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$$\rho_n = \sqrt[n]{r}\left(\cos\frac{\theta+2(n-1)\pi}{n} + i\sin\frac{\theta+2(n-1)\pi}{n}\right) = \sqrt[n]{r}e^{i(\frac{\theta+2(n-1)\pi}{n})};\ (\text{putting } k = n - 1)$$

All of the n roots will lie on a circle of radius $|\sqrt[n]{r}|$ in the complex plane and angular separation between two consecutive roots will be $\frac{2\pi}{n}$.

Note:

In real number system we can find out roots of only positive numbers and can find out certain roots of certain numbers with ease.

For example, we can find out cube root of 8 easily but not fifth root of 8 or (-8).

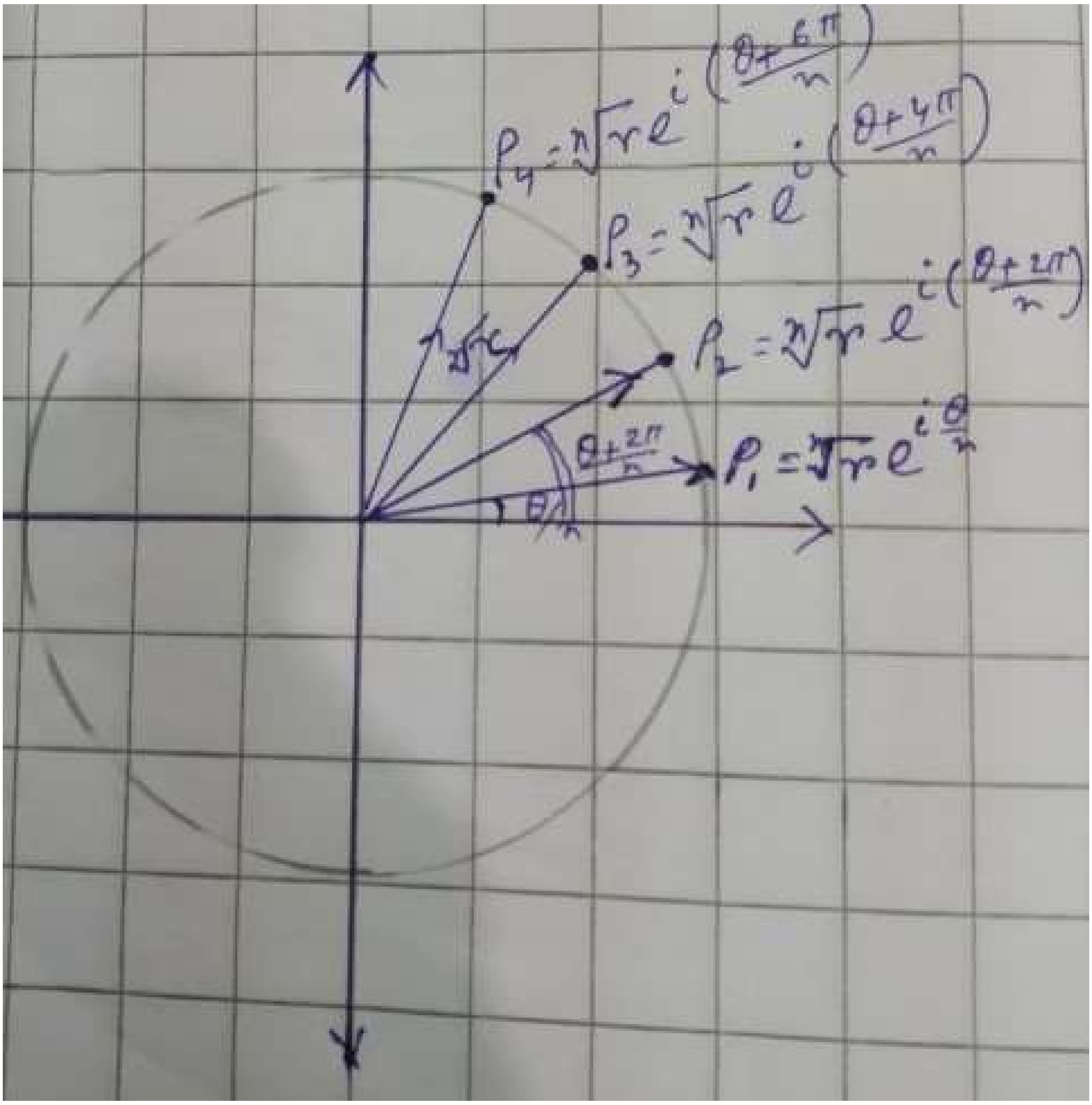
But in complex number system we can find out any root of any number, real or imaginary, positive or negative just by following a general prescription as follows:

Step 1: Express the given number in its polar form, ($z = re^{i\theta}$),

Step 2: take nth root of r ($i.e., \sqrt[n]{r}$) and divide $(\theta + 2\pi k)$ by *n*.

Step 3: Now draw a circle of radius $\sqrt[n]{r}$. Spot the first root as a point on the circle making an angle $\frac{\theta}{n}$ with the +ve real number axis. Then spot the second and subsequent roots as points on the circle in sequence of increasing angle, symmetrically spaced but $\frac{2\pi}{n}$ angle apart from each other.

Step 4: express the roots in Cartesian form, in $(x + iy)$. These are our desired roots.



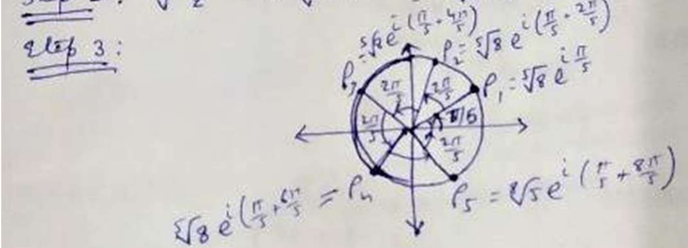
Example: Following the above prescription let us try to find the fifth root of (-8).

Solution: step-1: Let $z = -8 = -8 + i.0 = 8e^{i\pi}$

We have to find out $\sqrt[5]{z} = \sqrt[5]{-8} = \sqrt[5]{8e^{i\pi}}$

Step-2: $\sqrt[5]{z} = \sqrt[5]{8e^{i\pi}} = \sqrt[5]{8}\, e^{i\left(\frac{\pi + 2k\pi}{5}\right)}$

Step 3:



Step 4: So, roots are, $\rho_1 = \sqrt[5]{8}\, e^{i\frac{\pi}{5}} = \sqrt[5]{8}\left(\cos 36^\circ + i \sin 36^\circ\right)$

$\rho_2 = \sqrt[5]{8}\left(\cos 108^\circ + i \sin 108^\circ\right)$

$\rho_3 = \sqrt[5]{8}\left(\cos 180^\circ + i \sin 180^\circ\right) = -\sqrt[5]{8}$

$\rho_4 = \sqrt[5]{8}\left(\cos 252^\circ + i \sin 252^\circ\right)$

$\rho_5 = \sqrt[5]{8}\left(\cos 324^\circ + i \sin 324^\circ\right)$

Following above procedure let us do some more problems (From "Mathematical Methods in the Physical Sciences" 3rd edition by Mary L. Boas, page 66)

Problem set 1:

a. Plot and find all the values of the indicated roots:
1. $\sqrt[3]{1}$, 2. $\sqrt[3]{27}$, 3. $\sqrt[4]{1}$, 4. $\sqrt[4]{16}$, 5. $\sqrt[6]{1}$, 6. $\sqrt[6]{64}$,
7. $\sqrt[8]{16}$, 8. $\sqrt[8]{1}$, 9. $\sqrt[5]{1}$, 10. $\sqrt[5]{32}$, 11. $\sqrt[3]{-8}$, 12. $\sqrt[3]{-1}$,
13. $\sqrt[3]{i}$, 14. $\sqrt[3]{-8i}$, 15. $\sqrt{2 + 2i\sqrt{3}}$,, 16. $\sqrt[3]{2i - 2}$, 17. $\sqrt[4]{8i\sqrt{3} - 8}$,
18. $\sqrt[5]{-1 - i}$, 19. $\sqrt[5]{i}$,

b. Show that sum of the three cube roots of 8 is zero.

c. Show that sum of all the n nth roots of any complex number is zero

Solution of problem e

Let the complex number be $z = re^{i\theta}$

Hence n no. of nth roots would be given by

$$\rho_n = (r)^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \quad ; \text{ for } k = 0, 1, 2, \ldots, (n-1)$$

$$\therefore \quad \rho_1 = (r)^{1/n} e^{i\frac{\theta}{n}}$$

$$\rho_2 = (r)^{1/n} e^{i\left(\frac{\theta + 2\pi}{n}\right)}$$

$$\vdots$$

$$\rho_n = (r)^{1/n} e^{i\left[\frac{\theta + 2(n-1)\pi}{n}\right]}$$

$\therefore$ Sum of all the n roots would be

$$S = \rho_1 + \rho_2 + \rho_3 + \cdots + \rho_n$$

$$= r^{1/n} e^{i\frac{\theta}{n}} + r^{1/n} e^{i\left(\frac{\theta+2\pi}{n}\right)} + r^{1/n} e^{i\left(\frac{\theta+4\pi}{n}\right)} + \cdots + r^{1/n} e^{i\left[\frac{\theta+2(n-1)\pi}{n}\right]}$$

$$= r^{1/n} e^{i\frac{\theta}{n}} \left[1 + e^{i\frac{2\pi}{n}} + e^{i\frac{4\pi}{n}} + e^{i\frac{6\pi}{n}} + \cdots + e^{i\frac{2\pi(n-1)}{n}} \right]$$

$$= r^{1/n} e^{i\frac{\theta}{n}} \left[1 + x + x^2 + x^3 + \cdots + x^{(n-1)} \right], \text{ where } x = e^{i\frac{2\pi}{n}}$$

$$= r^{1/n} e^{i\frac{\theta}{n}} \left[\frac{1.(1 - x^n)}{1 - x} \right]$$

$$= r^{1/n} e^{i\frac{\theta}{n}} \left[\frac{1.(1 - e^{i\left(\frac{2n\pi}{n}\right)})}{1 - e^{i\frac{2\pi}{n}}} \right]$$

$$= r^{1/n} e^{i\frac{\theta}{n}} \left[\frac{1 - e^{i2\pi}}{1 - e^{i\frac{2\pi}{n}}} \right]$$

$$= r^{1/n} e^{i\frac{\theta}{n}} \left[\frac{1 - 1}{1 - e^{i\frac{2\pi}{n}}} \right]$$

$$= 0$$

In our next class we shall study:

Functions of Complex Variables. Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions.